Left Hopf algebroids, (quasi)-Frobenius algebras

Sophie Chemla

Sorbonne Université

London, July 2023

For Hopf algebra, the notion of integral was introduced by Sweedler (1969) and Larson-Sweedler (1969) proved their theorem for Hopf modules.
What is a Hopf algebra over a non necessarily basis?

- Hopf algebroids in the sense of Böhm (Lu, Böhm-Szlachanyi , etc...) : An antipode is assumed to exist. Integral theory was studied by Böhm (2005).
- $\times_A$-Hopf algebras (in the sense of Schauenburg) or left Hopf algebroids. An antipode is not required to exist but for any element h, the element $h_{(1)} \otimes S(h_2)$. Can be seen as Hopf monads and integral theory was developped by Bruguières-Virelizier (2007).

Hopf algebroids are left Hopf algebroids but the converse is not true in general (see Krähmer-Rovi 2015).

We will extend some results of Böhm to left Hopf algebroids thanks to a recent result of Schauenburg (explicit formulas given by Kowalzig). We will characterize left Hopf algebroids that are a (quasi)-Frobenius extension of their basis.
Many author have studied relations between Hopf algebras and Frobenius algebras : Pareigis, Böhm-Nill-Szlachányi, Böhm, Iovanov-Kadison, Balan, Saracco, etc...

- k will be a field and A will be a k-algebra with unit. Unadorned tensor products are tensor products over k.
- An A-ring (H, μ, η) is a monoid in the monoidal category $(A^e\text{-Mod}, \otimes_A, A)$ of A^e-modules fulfilling the associativity and the unitarity conditions.
- (Bohm) A–rings H correspond bijectively to k–algebra homomorphisms $\iota : A \longrightarrow H$. An A-ring H is endowed with an A^e-module structure:

$$\forall h \in H, \quad a, b \in H, \quad a \cdot h \cdot b = \iota(a) h \iota(b).$$

- An A-coring C is a comonoid in the monoidal category of A^e-modules satisfying the coassociativity and the counitarity conditions. As usual, we adopt Sweedler's Σ–notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$ or $\Delta(c) = c^{(1)} \otimes c^{(2)}$ for $c \in C$.

For an $A^e = A \otimes A^{op}$-ring U given by the k-algebra morphism $\eta : A^e \to U$, consider the restrictions

$$s := \eta(- \otimes 1_U) : A \to U \text{ and } t := \eta(1_U \otimes -) : A^{op} \to U,$$

called *source* and *target* map, respectively. Thus an A^e-ring U carries two A-module structures from the left and two from the right, namely

$$a \triangleright u \triangleleft b := s(a)t(b)u, \qquad a \blacktriangleright u \blacktriangleleft b := ut(a)s(b), \ \forall\ a, b \in A, u \in U.$$

If we let $U_{\triangleleft} \otimes_{A \triangleright} U$ be the corresponding tensor product of U (as an A^e-module) with itself, we define the *(left) Takeuchi-Sweedler product* as

$$U_{\triangleleft} \times_{A \triangleright} U := \left\{ \textstyle\sum_i u_i \otimes u_i' \in U_{\triangleleft} \otimes_{A \triangleright} U \mid \sum_i (a \blacktriangleright u_i) \otimes u_i' = \sum_i u_i \otimes (u_i' \blacktriangleleft a),\ \forall a \in A \right\} \tag{0.1}$$

By construction, $U_{\triangleleft} \times_{A \triangleright} U$ is an A^{e}-submodule of $U_{\triangleleft} \otimes_{A \triangleright} U$; it is also an A^e-ring via factorwise multiplication, with unit $1_U \otimes 1_U$ and $\eta_{U_{\triangleleft} \times_{A \triangleright} U}(a \otimes \tilde{a}) := s(a) \otimes t(\tilde{a})$.

For an $A^e = A \otimes A^{op}$-ring U given by the k-algebra morphism $\eta : A^e \to U$, consider the restrictions

$$s := \eta(- \otimes 1_U) : A \to U \text{ and } t := \eta(1_U \otimes -) : A^{op} \to U,$$

called *source* and *target* map, respectively. Thus an A^e-ring U carries two A-module structures from the left and two from the right, namely

$$a \triangleright u \triangleleft b := s(a)t(b)u, \qquad a \blacktriangleright u \blacktriangleleft b := ut(a)s(b), \ \forall\, a, b \in A, u \in U.$$

If we let $U_{\triangleleft} \otimes_{A \triangleright} U$ be the corresponding tensor product of U (as an A^e-module) with itself, we define the *(left) Takeuchi-Sweedler product* as

$$U_{\triangleleft} \times_{A \triangleright} U := \left\{ \textstyle\sum_i u_i \otimes u'_i \in U_{\triangleleft} \otimes_{A \triangleright} U \mid \sum_i (a \blacktriangleright u_i) \otimes u'_i = \sum_i u_i \otimes (u'_i \blacktriangleleft a),\ \forall a \in A \right\} \tag{0.1}$$

By construction, $U_{\triangleleft} \times_{A \triangleright} U$ is an A^{e}-submodule of $U_{\triangleleft} \otimes_{A \triangleright} U$; it is also an A^e-ring via factorwise multiplication, with unit $1_U \otimes 1_U$ and $\eta_{U_{\triangleleft} \times_{A \triangleright} U}(a \otimes \tilde{a}) := s(a) \otimes t(\tilde{a})$.
Can also define the right Takeuchi-Sweedler product as $U_{\blacktriangleleft} \times_{A \blacktriangleright} U$, which is an A^e-ring inside $U_{\blacktriangleleft} \otimes_{A \blacktriangleright} U$.

Definition

(Takeuchi) A *left bialgebroid* (U, A) is a k-module U with the structure of an A^{e}-ring (U, s^{ℓ}, t^{ℓ}) and an A-coring $(U, \Delta_{\ell}, \epsilon)$ subject to the following compatibility relations:

1. the A^{e}-module structure on the A-coring U is that of ${}_{\triangleright}U_{\triangleleft}$;
2. the coproduct Δ_{ℓ} is a unital k-algebra morphism taking values in $U_{\triangleleft} \times_{A} {}_{\triangleright}U$;
3. for all $a, b \in A$, $u, u' \in U$, one has $\epsilon(1_U) = 1_A$ and :

$$\epsilon(a \triangleright u \triangleleft b) = a\epsilon(u)b, \ \epsilon(uu') = \epsilon\big(u \blacktriangleleft \epsilon(u')\big) = \epsilon\big(\epsilon(u') \blacktriangleright u\big). \quad (0.2)$$

A *morphism* between left bialgebroids (U, A) and (U', A') is a pair (F, f) of maps $F : U \to U'$, $f : A \to A'$ that commute with all structure maps in an obvious way.

Remark

Szlachànyi has shown that left bialgebroids may be interpreted in terms of bimonads.

A *morphism* between left bialgebroids (U, A) and (U', A') is a pair (F, f) of maps $F : U \to U'$, $f : A \to A'$ that commute with all structure maps in an obvious way.

Remark

Szlachànyi has shown that left bialgebroids may be interpreted in terms of bimonads.

The notion of a *right bialgebroid* is obtained from that of *left bialgebroid* exchanging the role of $\triangleright, \triangleleft$ and $\blacktriangleright, \blacktriangleleft$. Then one starts with the A^{e}-module structure given by $\blacktriangleright$ and $\blacktriangleleft$ instead of $\triangleright$ and $\triangleleft$ and the coproduct takes values in $U_{\blacktriangleleft} \times_{A} {}_{\blacktriangleright}U$ instead of $U_{\triangleleft} \times_{A} {}_{\triangleright}U$. We refer to Kadison-Szlachanyi for details.

Remark

The *opposite* of a left bialgebroid $(U, A, s^{\ell}, t^{\ell}, \Delta_{\ell}, \epsilon)$ yields a *right* bialgebroid $(U^{\mathrm{op}}, A, t^{\ell}, s^{\ell}, \Delta_{\ell}, \epsilon)$. The *coopposite* of a left bialgebroid is the *left* bialgebroid given by $(U, A^{\mathrm{op}}, t^{\ell}, s^{\ell}, \Delta_{\ell}^{\mathrm{coop}}, \epsilon)$.

Definition

(Schauenburg) A left bialgebroid U is called a *left Hopf algebroid* or $\times_A$ *Hopf algebra* if the Hopf Galois map α_ℓ

$$\alpha_\ell : {}_{\blacktriangleright}U \otimes_{A^{\rm op}} U_{\triangleleft} \;\to\; U_{\triangleleft} \otimes_A {}_{\triangleright}U, \quad u \otimes_{A^{\rm op}} v \;\mapsto\; u_{(1)} \otimes_A u_{(2)} v,$$

is a bijection. We adopt for all $u \in U$ the following (Sweedler-like) notation

$$u_+ \otimes_{A^{\rm op}} u_- := \alpha_\ell^{-1}(u \otimes_A 1) \tag{0.3}$$

and the map $u \mapsto u_+ \otimes_{A^{\rm op}} u_-$ is called the *translation maps*.

Example

If $A = k$, U is a left Hopf algebroid if and only if U is a Hopf algebra and $u_+ \otimes u_- = u_{(1)} \otimes S(u_{(2)})$.

Definition

Likewise, U is called a *left opHopf algebroid* if the Galois map α_r is a bijection.

$$\alpha_r : U_{\blacktriangleleft} \otimes^{A} {}_{\triangleright}U \quad \to \quad U_{\triangleleft} \otimes_{A} {}_{\triangleright}U, \quad u \otimes^{A} v \quad \mapsto \quad u_{(1)} v \otimes_{A} u_{(2)}.$$

We set

$$u_{[+]} \otimes^{A} u_{[-]} := \alpha_r^{-1}(1 \otimes_{A} u), \tag{0.4}$$

and the map $u \mapsto u_{[+]} \otimes^{A} u_{[-]}$ is called *translation maps*.

Definition

Likewise, U is called a *left opHopf algebroid* if the Galois map α_r is a bijection.

$$\alpha_r : U_{\blacktriangleleft} \otimes^{A} {}_{\triangleright}U \quad \to \quad U_{\triangleleft} \otimes_{A} {}_{\triangleright}U, \quad u \otimes^{A} v \quad \mapsto \quad u_{(1)}v \otimes_{A} u_{(2)}.$$

We set

$$u_{[+]} \otimes^{A} u_{[-]} := \alpha_r^{-1}(1 \otimes_{A} u), \tag{0.4}$$

and the map $u \mapsto u_{[+]} \otimes^{A} u_{[-]}$ is called *translation maps*.

Example

If $A = k$, U is a left opHopf algebroid if and only if U_{coop} is a Hopf algebra and $u_{[+]} \otimes u_{[-]} = u_{(2)} \otimes S^{-1}(u_{(1)})$.

Definition

Let W be a *right* B-bialgebroid. Then W is called a *right Hopf algebroid (=RHB)*, respectively a *right opHopf algebroid (=RopHB)* if the Galois maps

β_r, resp. β_ℓ, is a bijection.

$$\beta_\ell : W_{\triangleleft} \otimes_B {}_{\blacktriangleright}W \to W_{\blacktriangleleft} \otimes_B {}_{\blacktriangleright}W, \quad w \otimes y \mapsto yw^{(1)} \otimes w^{(2)},$$
$$\beta_r : {}_{\blacktriangleright}W \underset{B^{op}}{\otimes} W_{\triangleleft} \to W_{\blacktriangleleft} \underset{B}{\otimes} {}_{\blacktriangleright}W, \qquad w \otimes y \mapsto w^{(1)} \otimes y\, w^{(2)}.$$

In either case, we adopt the following (Sweedler-like) notation:

$$w^- \otimes w^+ := \beta_r^{-1}(w \otimes 1), \qquad w^{[-]} \otimes w^{[+]} := \beta_l^{-1}(1 \otimes w), \ \forall \ w \in W,$$

for the translation maps.

The example provide by Lie Rinehart algebras

In this example A will be a commutative k-algebra. The vector space of derivations of A, $Der(A)$, is endowed with a natural A-module structure.

Definition

(Rinehart 1962) A Lie Rinehart algebra (or Lie algebroid) over A is a triple $(L, [-,-], \rho)$ where

- $[-,-] : L \times L \to L$ is a k-Lie algebra
- L is a (finitely generated projective) A-module
- $\rho : L \to Der(A)$ (the anchor) is an A-module morphism and a Lie algebra morphism.
- $\forall X, Y \in L, \quad \forall a \in A,$

$$[X, aY] = \rho(X)(a)Y + a[X, Y].$$

Examples

Example 1: $L = Der(A)$ and $\rho = id$.
Example 2 : $A = k$. Then $\rho = 0$ and L is a k-Lie algebra.

Examples

Example 1: $L = Der(A)$ and $\rho = id$.
Example 2 : $A = k$. Then $\rho = 0$ and L is a k-Lie algebra.
Example 3: $(M, \{-,-\})$ is a Poisson manifold with Poisson bivector $\pi \in \Gamma(M, \wedge^2 TM)$, the $A = \mathcal{C}^\infty(M)$-module of global differential one forms $\Gamma(T^*M)$ is endowed with a Lie Rinehart algebra structure over A as follows :

- The anchor $\rho : \Gamma(T^*M) \to \Gamma(TM)$ is the map defined by π.
- If ω_1 and ω_2 are two global one forms

$$[\omega_1, \omega_2] = L_{\pi^\sharp(\omega_1)}(\omega_2) - L_{\pi^\sharp(\omega_2)}(\omega_1) + \pi(\omega_1, \omega_2).$$

More algebraically: For any $a, b, u, v \in A$,

- $[adu, bdv] = a\{u, b\}dv - b\{v, a\}du + abd\{u, v\}$
- $\rho(adu) = a\{u, -\}$.

To a Lie Rinehart algebra is associated its enveloping algebra

$$U_A(L) = \frac{T_k^+(A \oplus L)}{J}$$

where J is the two sided ideal generated by the relations: For all $a, b \in A$, for all $D, \Delta \in L$,

1. $a \otimes b - ab$
2. $D \otimes \Delta - \Delta \otimes D - [D, \Delta]$
3. $D \otimes a - a \otimes D - \rho(D)(a)$
4. $a \otimes D - aD$

Examples

1. If $A = k$, L is a Lie algebra and we recover the enveloping algebra of a Lie algebra.
2. If M is a $\mathcal{C}^\infty$-manifold and $L = \Gamma(TM)$ the enveloping algebra of the Lie Rinehart algebra $(\Gamma(TM), id)$ is the algebra of globally defined differential operators.

(Rinehart 1962) PBW theorem holds for $U_A(L)$ if the A-module L is projective.

If L is a $k - A$- Lie Rinehart algebra, $U_A(L)$ is endowed with a standard left bialgebroid structure as follows (Xu):

1. For all $a \in A$, $s^\ell(a) = t^\ell(a) = a$
2. The coproduct Δ is defined by

$$\forall a \in A, \quad \Delta(a) = a \otimes 1, \qquad \forall D \in L, \quad \Delta(D) = D \otimes 1 + 1 \otimes D$$

3. $\epsilon(D) = 0$ and $\epsilon(a) = a$.

Moreover, $U_A(L)$ is a left Hopf algebroid. The translation maps is determined by the equalities: For all $a \in A$ and all $D \in L$.

$$\begin{aligned} a_+ \otimes a_- &= a \otimes 1 \\ D_+ \otimes D_- &= D \otimes 1 - 1 \otimes D. \end{aligned}$$

As $U_A(L)$ is cocommutative, it is also (op)Hopf.

If L is a $k-A$- Lie Rinehart algebra, $U_A(L)$ is endowed with a standard left bialgebroid structure as follows (Xu):

1. For all $a \in A$, $s^{\ell}(a) = t^{\ell}(a) = a$
2. The coproduct Δ is defined by

$$\forall a \in A, \quad \Delta(a) = a \otimes 1, \qquad \forall D \in L, \quad \Delta(D) = D \otimes 1 + 1 \otimes D$$

3. $\epsilon(D) = 0$ and $\epsilon(a) = a$.

Moreover, $U_A(L)$ is a left Hopf algebroid. The translation maps is determined by the equalities: For all $a \in A$ and all $D \in L$.

$$\begin{aligned} a_+ \otimes a_- &= a \otimes 1 \\ D_+ \otimes D_- &= D \otimes 1 - 1 \otimes D. \end{aligned}$$

As $U_A(L)$ is cocommutative, it is also (op)Hopf.
If L is projective, $U_A(L)$ satisfies the PBW theorem (see Rinehart) .
It is a projective A-module but not finitely generated. To get a finitely generated projective left Hopf algebroid, one can take k of characteristic p and take the restricted enveloping algebra $U'_A(L)$.

Let (U, A) be a left bialgebroid. We set

$$U_* := \mathrm{Hom}_A({}_{\triangleright}U, A) \text{ and } U^* := \mathrm{Hom}_{A^{\mathrm{op}}}(U_{\triangleleft}, A),$$

called, respectively, the *left* and *right* dual of U.
The two dual are endowed with an A^e-ring structure, and even a right bialgebroid structure under finiteness and projectiveness conditions (Kadison-Szlachanyi).

The case of U^*:

For $a \in A$, let us introduce the two elements $s_r^*(a)$ and $t_r^*(a)$ of U^* defined by

$$\forall u \in U, \quad < t_r^*(a), u >= a < \epsilon, u >, \quad < s_r^*(a), u >=< \epsilon, us^\ell(a) > . \tag{0.5}$$

Endowed with the following multiplication, U^* is an associative k-algebra with unit ϵ: For all $\phi, \phi' \in U^*$ and all $u \in U$

$$\Big\langle\, u\,,\, \phi\,\phi' \Big\rangle \;=\; \Big\langle\, s^\ell\big(\big\langle\, u_{(1)}\,,\, \phi\,\big\rangle\big)\, u_{(2)}\,,\, \phi' \,\Big\rangle \tag{0.6}$$

Then $s_r^* : A \to U^*$ and $t_r^* : A^{op} \to U^*$ are algebra morphisms and define an A^e-ring structure on U^*:

$$\phi \blacktriangleleft a = \phi s_r^*(a) \quad \text{and} \quad a \blacktriangleright \phi = \phi t_r^*(a).$$

The product on U^* can be written :

$$\Big\langle\, u\,,\, \phi\,\phi' \Big\rangle \;= \Big\langle\, u_{(2)}\,,\, t_r^*\big(\big\langle\, u_{(1)}\,,\, \phi\,\big\rangle\big)\, \phi' \,\Big\rangle \tag{0.7}$$

If $U_{\triangleleft}$ is a finite projective A^{op}-module, the following formula defines a coproduct on U^* :

$$\left\langle u\, u' \,,\, \phi \right\rangle \;=\; \left\langle u\, t_\ell\big(\langle u' \,,\, \phi_{(2)} \rangle\big) \,,\, \phi_{(1)} \right\rangle \;=\; \left\langle u \,,\, \phi_{(1)}\, s_r^*\big(\langle u' \,,\, \phi_{(2)} \rangle\big) \right\rangle$$

Lastly we have a counit $\eta \in U^*$

$$\left\langle 1 \,,\, \phi \right\rangle \;=\; \eta(\phi) \tag{0.8}$$

Thus $(U^*, A, s_r^*, t_r^*, \Delta, \eta)$ is a right bialgebroid.

If $U_{\triangleleft}$ is a finite projective A^{op}-module, the following formula defines a coproduct on U^* :

$$\left\langle u\, u' \,,\, \phi \right\rangle \;=\; \left\langle u\, t_\ell\big(\langle u' \,,\, \phi_{(2)} \rangle\big) \,,\, \phi_{(1)} \right\rangle \;=\; \left\langle u \,,\, \phi_{(1)}\, s_r^*\big(\langle u' \,,\, \phi_{(2)} \rangle\big) \right\rangle$$

Lastly we have a counit $\eta \in U^*$

$$\left\langle 1 \,,\, \phi \right\rangle \;=\; \eta(\phi) \tag{0.8}$$

Thus $(U^*, A, s_r^*, t_r^*, \Delta, \eta)$ is a right bialgebroid.

The case of U_*: If ${}_{\triangleright}U$ is a finite projective A-module, U_* is endowed with the right bialgebroid structure over A such that $(U_{coop})_* = (U^*)_{coop}$.

Similarly, if W is a right bialgebroid over A, its left dual ${}_*W$ and its right dual ${}_*W$ are endowed with left bialgebroid structure over A.

Similarly, if W is a right bialgebroid over A, its left dual ${}_*W$ and its right dual ${}_*W$ are endowed with left bialgebroid structure over A.

Theorem

(Schauenburg (2017), explicit formulas by Kowalzig)
If U is a left Hopf algebroid, then U^* (respectively U_*) is a right (op)Hopf algebroid.

Hopf modules

Left-left Hopf modules are the objects of study of the fundamental theorem for Hopf modules (Larson-Sweedler). The latter states that, if H is a k-Hopf algebra, there is an equivalence of categories between left-left Hopf modules and k-vecteor spaces. Left-left Hopf modules can be defined in the case of Hopf algebroids (in the sense of Böhm), in the framework of bimonads over a monoidal category (Bruguières-Virelizier) and in the context of Hopf categories (Batista-Caenepeel-Vercruysse). In all these cases, the Larson-Sweedler theorem for Hopf modules was proved. We will use only a part of this theorem that follows from a flat descent argument (due to Brzezinski).

Definition

1) Let $(W, B, s^r, t^r, \Delta, \partial)$ be a right bialgebroid over the k-algebra A. We will say that M is endowed with a right-right Hopf W-module structure if

- (i) M is endowed with a right W-module structure.
- (ii) M is endowed with a right W-comodule structure denoted Δ_M.
- (iii) These two structures are linked by the following relation : for all $m \in M$, $w \in W$ and $b \in B$

$$m_{(0)}w_{(1)} \otimes m_{(1)}w_{(2)} = \Delta_M(mw).$$

(iv) $m \cdot b = ms^r(b)$.

2) Left left Hopf modules are defined over a left bialgebroid.

Example

If N is a right A-module, then $N \otimes_{A_\blacktriangleright} W$ is a right right Hopf W-module as follows: For all $(w, v) \in W^2$ and all $n \in N$,

$$(n \otimes_{A_\blacktriangleright} w) \cdot v = n \otimes_{A_\blacktriangleright} wv \quad and \quad \Delta_{P \otimes_A W}(n \otimes w) = n \otimes w_{(1)} \otimes w_{(2)}$$

It follows from the fundamental theorem for Hopf modules (Larson-Sweedler for Hopf algebras, Böhm for Hopf algebroids, Bruguières-Virelizier for Hopf monads, Batista-Caenepeel-Vercrruysse for Hopf categories, etc...), that : if W is a right Hopf algebroid and under flatness conditions, all right right Hopf W-modules are of this type (up to isomorphisms).

Theorem

([C]) Let U be a left Hopf left bialgebroid such that $U_{\triangleleft}$ is a finitely generated projective A^{op}-module, then U^* is a right Hopf algebroid with translation map

$$\phi \in U^* \mapsto \phi^- \otimes \phi^+ \in_{\blacktriangleright} U^* \otimes_{A^{op}} U^*_{\triangleleft}.$$

If $\phi \in U^*$ and $u \in U$,

$$\begin{array}{l} u \cdot \phi = \epsilon_r^{U^*}\left[t_r^*\left(< u, \psi\phi^- > \phi^+\right)\right] \\ \Delta_U(u) = e_i u_{\triangleleft} \otimes_{A_{\blacktriangleright}} e_i^* \end{array}$$

where (e_i, e_i^*) is the dual basis of the right finitely generated A^{op}-module $U_{\triangleleft}$.

From the Larson-Sweedler theorem for Hopf modules, we deduce an isomorphism of right U^*-modules and right U^*-comodules

$$\begin{array}{rcl} {}_{\blacktriangleright}U^{cov} \otimes_A U^*_{\triangleleft} & \simeq & U \\ u_0 \otimes \phi & \mapsto & u_0 \cdot \phi \end{array}$$

But $U^{cov} = \{u \in U, \quad \forall v \in U,\ uv = s^{\ell}\epsilon(u)v\}$ is the A-module of left integrals of U.

Remark

Case of Hopf algebras (Larson-Sweedler), case of Hopf algebroid (Böhm).

Frobenius extensions

A monomorphism of k-algebras $s : A \to U$ defines an A^e-module structure on U : Forall $(a, b) \in A^2$, $u \in U$,

$$a \cdot u \cdot b = s(a)us(b).$$

As usual, $a \cdot u \cdot b$ will be denoted $a \triangleright u \blacktriangleleft b$. Recall that an A^e-module structure on U defines an A^e-module structure on U_* as follows : Forall $\psi \in U_*$, $a \in A$, $v \in U$,

$$a \blacktriangleright \psi = s(a) \rightharpoonup \psi, \quad < \psi \blacktriangleleft a, v >=< \psi, v > a.$$

It is also endowed with the left U-module structure given by the transpose of the right multiplication

$$\forall \psi \in U_*, \quad \forall (u, v) \in U^2, \quad (v \rightharpoonup \psi)(u) = \psi(uv).$$

Definition

(Karsch 1954) A monomorphism of k-algebras $s : A \to U$ is called a Frobenius extension if

1. $_{\triangleright}U$ is finitely generated and projective
2. The $U \otimes A^{op}$-modules $_{U}U_{\blacktriangleleft}$ and $U_{*\blacktriangleleft}$ are isomorphic

Proposition

([C]) Let $(U, A, s^\ell, t^\ell, \Delta^\ell, \epsilon)$ *be a left Hopf algebroid such that the* A^{op}*-module* $U^*{}_\triangleleft$ *is flat. The extension* $t^\ell : A^{op} \to U$ *is Frobenius if and only if*

1. $U_\triangleleft$ *is a finitely projective* A^{op}*-module*
2. ${}_\blacktriangleright\left(\int_U^\ell\right)$ *is a free A-module of rank 1.*

The proof follows from the Larson-Sweedler theorem for Hopf modules.

Remarks

1. If $A = k$ is a field, the k-algebra U is Frobenius if and only if the monomorphism $k \to U$ is a Frobenius extension.
2. Pareigis showed that a A-Hopf algebra (with A commutative) satisfying the two conditions of the theorem is Frobenius.
3. Böhm : Case of Hopf algebroid:
4. M.C. Iovanov and L. Kadison investigated when a weak Hopf algebra is Frobenius.
5. Morita showed that the monomorphism $s : A \to U$ is a Frobenius extension if and only if the restriction functor is a Frobenius functor.

Definition

(Muller 1971) Recall that an A^e-module structure on U defines an A^e-module structure on U_* as follows : For all $\psi \in U_*$, $a \in A, v \in U$,

$$a \blacktriangleright \psi = s(a) \rightharpoonup \psi, \quad < \psi \blacktriangleleft a, v > = < \psi, v > a.$$

Endow U_* with the left U-module structure given by the transpose of the right multiplication

$$\forall \psi \in U_*, \quad \forall (u, v) \in U^2, \quad (v \rightharpoonup \psi)(u) = \psi(uv).$$

A monomorphism of k-algebras $s : A \to U$ is called quasi-Frobenius if

1. ${}_{\triangleright}U$ is finitely generated and projective
2. The $U \otimes A^{op}$-module ${}_U U_{\blacktriangleleft}$ is a direct summand in a finite direct sum of copies of $U_{*\blacktriangleleft}$.

Remarks

1. Quasi-Frobenius functors were introduced by Iglesias-Nastasescu-Vercruysse (2010). The monomorphism $s : A \to U$ is a quasi-Frobenius extension if and only is the restriction functor is a quasi Frobenius functor.
2. Pareigis (1964) showed that a finitely generated projective Hopf algebra over a commutative ring is quasi-Frobenius.
3. Böhm-Nill-Szlachányi showed that weak Hopf algebras are quasi-Frobenius.

Proposition

([C]) Let (U, A, s^ℓ, t^ℓ) *be a left Hopf algebroid such that the* A^{op}*-module* $U^*{}_\triangleleft$ *is flat. The extension* $t^\ell : A^{op} \to U$ *is quasi-Frobenius if and only if*

1. $U_\triangleleft$ *is a finitely projective* A^{op}*-module*
2. ${}_\blacktriangleright\left(\int_U^\ell\right)$ *is a finitely generated projective A-module.*

In this section, we apply our theory to the restricted enveloping algebra of a restricted Lie-Rinehart algebra. We will assume that k is a field of characteristic p.

Definition

Let A be a commutative k-algebra and let $(A, L, (-)^{[-]}, \omega)$ be a restricted Lie-Rinehart algebra. The restricted universal enveloping algebra $U'_A(L) = \dfrac{U_A(L)}{< D^p - D^{[p]}, \quad D \in L >}$.

We set $J'(L) = \left[U_A(L')\right]^*$.

Examples

1. If $A = k$, recover a restricted Lie algebra and its restricted enveloping algebra.
2. If $L = \mathrm{Der}(A)$, recover the algebra of differential operators over A.

Proposition

Assume that L restricted Lie-Rinehart algebra which is is a finitely generated projective A-module with a rank. Set $J'_A(L) = [U_A(L)']^*$*. Then* $\int^{\ell}_{U'_A(L)}$ *and* $\int^{\ell}_{J'_A(L)}$ *are projective A-module of rank one. Thus,* $s^{\ell} : A \to U'_A(L)$, $s^r_* : A \to J'_A(L)$ *and* $t^r_* : A \to J'_A(L)$ *are quasi-Frobenius extensions.*
They are Frobenius extension if L is a finitely generated free A-module.

Remark

The case of a Lie algebra had been proved by Berkson in 1964.

THANK YOU FOR YOUR ATTENTION!