The Hopf Algebroid of Partial Representations

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Hoids & NCG

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Outline

1. Motivation
2. Partial reps of groups
3. Partial reps of Hopf algebras
4. The algebra H_{par}
5. Hopf algebroid structure of H_{par}
6. Partial corepresentations

1 Motivation

geometry ⇝ NCG

Symmetries ⇝ groups + actions & representations ⇝ Hopf alg (co) action (co) representation

⇝

Partial Symmetries ⇝ partial action partial representation of groups ⇝ ___ ___ ___ of Hopf alg

↑

groupoids ⇝ Hopf algebroids

G group

k field (comm. ring)

V vector space / k

$\pi : G \longrightarrow \mathrm{End}_k(V)$

- $\pi(e_G) = \mathrm{id}_V$

$\Longrightarrow$

- $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1})$
- $\pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(g)\pi(h)$

Remarks: $\pi(g)$ is not always invertible.

$\pi(g)\pi(h) \neq \pi(gh)$

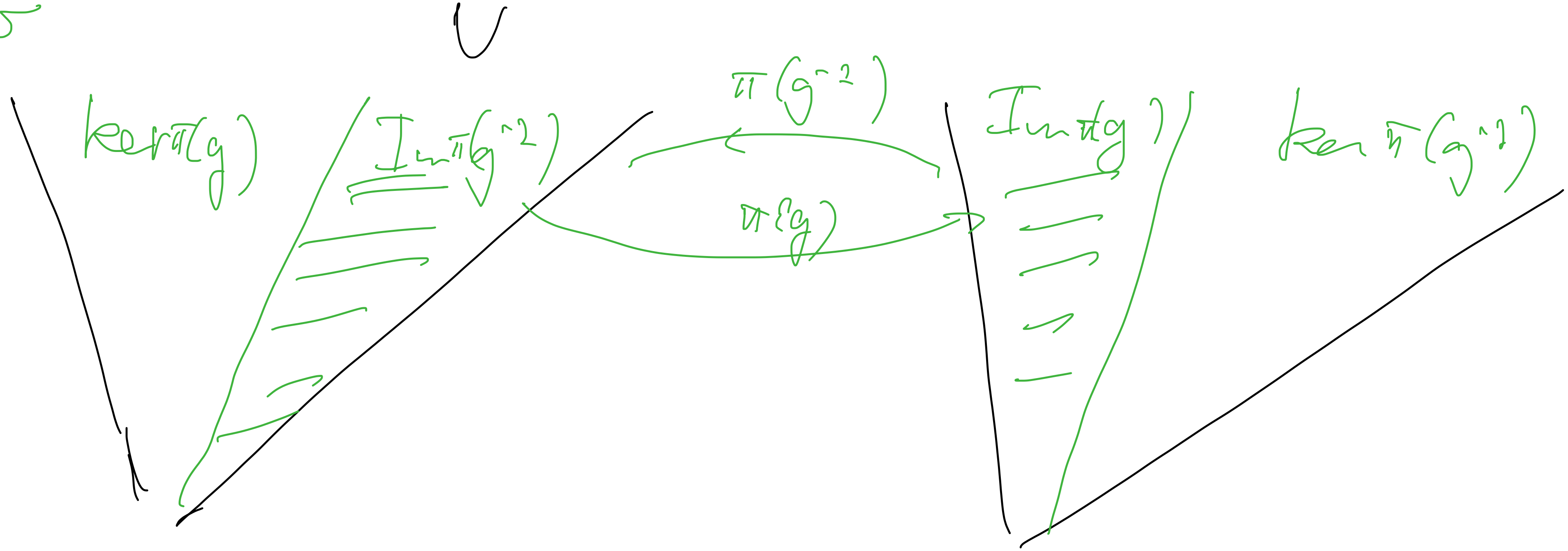
$\pi(g)$ is automorphism $\forall g \in G$ $\Leftrightarrow$ $\pi(g)\pi(g^{-1}) \neq \mathrm{id}_V$ $\forall g$ $\Leftrightarrow$ π being "global".

$$\pi(g)\,\pi(g^{-1})\,\pi(g) = \pi(g) \quad \forall\ g \in G.$$

$$\Rightarrow \quad \pi(g)\,\pi(g^{-1}) \in \mathrm{End}(V) \text{ is idempotent.}$$

$$V \simeq \ker \pi(g^{-1}) \oplus \mathrm{Im}\,\pi(g).$$

$g \in G$



③ Partial representations of Hopf algebras.

$$\pi : H \longrightarrow End_k(V)$$

$$\begin{cases} \cdot\ \pi(1_H) = id_V \\ \cdot\ \pi(h)\pi(g_{(1)})\pi(S(g_{(2)})) = \pi(hg_{(1)})\pi(S(g_{(2)})) \\ \cdot\ \pi(h_{(1)})\pi(S(h_{(2)}))\pi(g) = \pi(h_{(1)})\pi(S(h_{(2)})g) \end{cases}$$

⇝ $=0$ $\left|\ \begin{array}{l} \pi(h)\pi(S(g_1))\pi(g_2) = \pi(hS(g_1))\pi(g_2) \\ \pi(Sh_1)\pi(h_2)\pi(g) = \pi(Sh_1)\pi(h_2 g) \end{array}\right.$

P. Saracco.

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partial module: $H \otimes V \longrightarrow V$

$$h \otimes v \longmapsto \pi(h)(v) = h \cdot v$$

PMod^H

④ The algebra H_{par}.

$$ {}_{H}\mathrm{PMod} \cong {}_{H_{par}}\mathrm{Mod} $$

$$ H \to T(H)/I = H_{par}. $$

$$ h \longmapsto \langle [h] \mid h \in H \rangle \Big/ \langle [h][g_1][Sg_2] = [hg_1][Sg_2],\ [h_1][S(h_2)][g] = [h_1][S(h_2)g],\ [1_H] = 1_{H_{par}} \rangle $$

$$ H_{par} \xrightarrow{\text{rep.}} \mathrm{End}(V) $$

$[\ ] : H \to H_{par}$, $\quad H \to \mathrm{End}(V)$ $\ \forall \pi$ part. rep.

<u>Example</u> $H = k G \rightsquigarrow H_{par} = k_{par} G$

$\cong k\Gamma(G)$

groupoid alg.

$\Gamma(G)$ groupoid

- objects: $e \in A \subset G$
- arrows: $A \xrightarrow{g} B$

$$\boxed{\begin{array}{c} g^{-1} \in A \\ gA = B \end{array}}$$

$\rightarrow \mathbb{C}\Gamma(G) \cong \bigoplus_{H \leq G} \left(M_m(kH)\right)^{c_m}$ <u>SD.</u>

$\rightsquigarrow \mathbb{C}\mathbb{Z}_2 \cong \mathbb{C} \times \mathbb{C} \times \underline{\underline{\mathbb{C}}}$

Lie algebras. $H = U(\mathfrak{g})$

$${}_H\mathrm{Mod} = {}_H{}^P\mathrm{Mod}$$

$$H_{par} = H$$

Question

H s.s.

$\overset{?}{\Rightarrow}$ H_{par} s.s.??

H_4 Sweedler's 4-dim HA.

$H_{par} \rightsquigarrow$ infinite dim.

$H_4 = \langle 1, g, x, y = gx \rangle$ g grouplike.

$\Delta(x) = g \otimes x + x \otimes 1$

$W_n = k\langle w_1, \ldots, w_n \rangle$

$\pi(g) = 0.$

$\pi(1) = id$

$\pi(x)(w_k) = \begin{cases} w_{k+1} & k < n \\ 0 & k = n \end{cases}$

$W_1 \subset W_2 \subset W_3 \subset \ldots$

(5) H_{par} as Hopf algebroid. H bijective antipode

$$A_{par} \subset \underset{\tilde{A}_{par}}{H_{par}} = T(H) \Big/ \big\langle [h_1][S(h_2)][g] = [h_1][S(h_2)g], \ \ldots \big\rangle$$

$$A_{par} \cong \langle [h_1][S(h_2)] = \epsilon_h \mid h \in H \rangle$$

$$\tilde{A}_{par} = \langle [S(h_1)][h_2] \mid h \in H \rangle.$$

$$A_{par} \otimes A_{par}^{op} \longrightarrow H_{par}$$

H_{par} is A_{par} - Hopf algebroid.

$$\Delta : H_{par} \longrightarrow H_{par} \otimes_{A_{par}} H_{par}$$

$$[h^1] \cdots [h^n] \longrightarrow [h^1_{(1)}][h^2_{(1)}] \cdots [h^n_{(1)}] \otimes [h^1_{(2)}][h^2_{(2)}] \cdots [h^n_{(2)}]$$

$$\epsilon : H_{par} \longrightarrow A_{par}$$

$$[h] \longrightarrow [h_1][S(h_2)] = \epsilon_h$$

$$S : H_{par} \longrightarrow H_{par}$$

$$[h^1] \cdots [h^n] \longmapsto [S(h^n)] \cdots [S(h^1)].$$

$$(PMod, \otimes_{A_{par}})$$

$\underline{\mathrm{HAlg}} \longrightarrow \underline{\mathrm{HAlgoids}}$

$H \longmapsto H_{par}$

$\underline{\mathrm{Ex}}$ $\quad A_{par}(H_4) \cong k\langle x, z \rangle \big/ (2x^2 - x,\ 2xz - z)$

infinite dim

$(H_4)_{par}$ is $\underline{\underline{\text{not}}}$ a WHA.

⑥ Partial corepresentations over a HA H.

$$\rho : M \longrightarrow M \otimes H$$
$$m \longmapsto m^0 \otimes m^1$$

- $m^0 \epsilon(m^1) = m$.
- $m^{00} \otimes m^{01}{}_1 \otimes m^{01}{}_2 S(m^1)$

$$= m^{000} \otimes m^{001} \otimes m^{01} S(m^1)$$

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① Question

$\boxed{\mathrm{PMod}^H} \overset{??}{\not\simeq} \mathrm{Mod}^C$

for some coalg C??

do not satisfy the fundamental thm.

Ex : H_4

$k[2] \longrightarrow k[2] \otimes H_4$

$g(2^n) = 2^{n+1} \otimes y + \frac{1}{2} 2^n \otimes 1$

$+ \frac{1}{2} 2^n \otimes g.$

Solution 1

⇝ restrict to f.d. part. comodules

$PMod^{H}_{f.d.} \simeq Mod^{H^{par}_{Hopf}}$

↓ ⇝ Tannaka.

Vect

H^{par} coalg. ⇝ Hopf coalgebroid

∩ base coalgebra

Coefree (H) $H^{par} \overset{\text{dualize}}{\longleftrightarrow} (H^{o})_{par}$ C^{par}

Solution 2

Hopf ??

Grothendieck $\mathrm{PMod}^H \simeq \mathrm{Vect}^{\boxed{\mathbb{C}}}$

$\mathbb{C} = UR$ comonad

$U \downarrow \dashv \uparrow R$

$\mathrm{Vect}_k \rightleftarrows \mathrm{Vect}_k$

$(\mathcal{C}, \otimes) \rightsquigarrow \mathrm{Mod}^{C^{par}} \quad C^{par}$

??

Constructing R / $\mathbb{C}$ explicitly using topological coalg.

$$R(V) \subset \prod_{n \in \mathbb{N}} V \otimes H^{\otimes n}$$

Question ? is $\mathbb{C}$ Hopf

problem: ? over which monoidal base category