Hopf algebroids and noncommutative geometry

Queen Mary University of London — 12–14 July 2023

DUALITY FUNCTORS for QUANTUM GROUPOIDS

Fabio GAVARINI

Università di Roma "Tor Vergata"

— 13 July 2023 —

Journal of Noncommutative Geometry **9** (2015), no. 2, 287–358

joint work with Sophie CHEMLA (Paris "Sorbonne")

— BACKGROUND —

(1) — for each G and $\mathfrak{g} = \mathit{Lie}(G)$, $\exists$ quantum groups of two types, say "type $U_\hbar(\mathfrak{g})$" and "type $F_\hbar[[G]]$" — called QUEAs and QFSHAs

(2) — ***linear duality*** yields an **antiequivalence** $U_\hbar(\mathfrak{g}) \leftrightsquigarrow F_\hbar[[G]]$ between QUEAs and QFSHAs, "upon the *same* pair" $(G, \mathfrak{g})$

(3) — $\exists$ **equivalence** between QUEAs and QFSHAs which "lifts" the *Poisson duality* $(G, \mathfrak{g}) \leftrightsquigarrow \big(G^*, \mathfrak{g}^*\big)$ between pairs $(K, \mathfrak{k})$ of *Poisson* Lie groups & Lie *bi*algebras

— GOAL —

Replace the words "groups" & "Lie algebras" with "**groupoids** & **Lie algebroids**"...

...then try and achieve the same as **(1)–(2)–(3)** above for "**quantum groupoids**".

WHAT'S OLD: QUANTUM GROUP CASE

[1] — <u>**Classical setup:**</u> $G =$ (formal) Lie group, $\mathfrak{g} = Lie(G)$

$$(G, \mathfrak{g}) \xrightarrow[\text{description}]{\text{algebraic}} \begin{cases} U(\mathfrak{g}) & \text{Hopf algebra} \\ F[[G]] & \text{(topological) Hopf algebra} \end{cases}$$

FACT: $\exists$ *linear (Hopf) duality* yielding an **antiequivalence** $U(\mathfrak{g}) \leftrightsquigarrow F[[G]]$

i.e. $F[[G]] \cong U(\mathfrak{g})^*$ (*full* dual) & $U(\mathfrak{g}) \cong F[[G]]^\star$ (*topological* dual)

Poisson structures: recall that

G has a *Poisson* group structure i.e., $F[[G]]$ has a *Poisson* bracket	$\Longleftrightarrow$	$\mathfrak{g}$ has a Lie *bi*algebra structure i.e., $\mathfrak{g}$ has a Lie *co*bracket

Then ***Poisson duality*** holds, namely

$\mathfrak{g}$ is a Lie **bi**algebra $\Longleftrightarrow$ $\mathfrak{g}^*$ is a Lie **bi**algebra

and for (formal) Lie groups we have

G is a **Poisson** (formal) Lie group $\Longleftrightarrow$ G^* is a **Poisson** (formal) Lie group

[2] — Quantum setup (*à la Drinfeld*)

To $(G, \mathfrak{g})$ as above, we associate ***quantum groups*** as follows:

$U_\hbar(\mathfrak{g}) :=$ $\hbar$–adic (topological) Hopf algebra over $\Bbbk[[\hbar]]$ $\rightsquigarrow$ called *QUEA*s

$$\text{such that} \qquad U_\hbar(\mathfrak{g})\Big|_{\hbar=0} := U_\hbar(\mathfrak{g}) \Big/ \hbar\, U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})$$

$F_\hbar[[G]] :=$ $I_\hbar$–adic (topological) Hopf algebra over $\Bbbk[[\hbar]]$ $\rightsquigarrow$ called *QFSHA*s

$$\text{such that} \qquad F_\hbar[[G]]\Big|_{\hbar=0} := F_\hbar[[G]] \Big/ \hbar\, F_\hbar[[G]] \cong F[[G]]$$

FACT: $\exists$ *linear (Hopf) duality* yielding an **antiequivalence** $U_\hbar(\mathfrak{g}) \leftrightsquigarrow F_\hbar[[G]]$

i.e. $F_\hbar[[G]] \cong U_\hbar(\mathfrak{g})^*$ (*full* dual) & $U_\hbar(\mathfrak{g}) \cong F_\hbar[[G]]^\star$ (*topological* dual)

so $\big(\mathbf{QUEA}\big) \xleftrightarrow{\text{antiequivalence}} \big(\mathbf{QFSHA}\big)$ via $U_\hbar(\mathfrak{g}) \mapsto U_\hbar(\mathfrak{g})^*$ & $F_\hbar[[G]] \mapsto F_\hbar[[G]]^\star$

where $\big(\mathbf{QUEA}\big) :=$ category of all the QUEAs

and $\big(\mathbf{QFSHA}\big) :=$ category of all the QFSHAs

N.B.: this is a sheer "lifting" at the quantum level of the classical linear duality

[3] — Semiclassical structures: *Quantum* $\Longrightarrow$ *Poisson*

Every quantisation of $(G,\mathfrak{g})$ *defines a Poisson structure on the latter*, namely:

(F) — given $F_\hbar[[G]]$, a Poisson bracket $\{\ ,\ \} : F[[G]] \otimes F[[G]] \longrightarrow F[[G]]$ is defined on $F[[G]]$ by

$$\big\{ f, \ell \big\} \ := \ \frac{f'\,\ell' - \ell'\,f'}{\hbar} \ (\text{mod } \hbar) \qquad \forall\, f, \ell \in F[[G]], \quad \begin{array}{c} f', \ell' \in F_\hbar[[G]] \ : \\ f' \ (\text{mod } \hbar) \, = \, f \\ \ell' \ (\text{mod } \hbar) \, = \, \ell \end{array}$$

(U) — given $U_\hbar(\mathfrak{g})$, a Poisson cobracket $\delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is defined on $U(\mathfrak{g})$ — hence a Lie cobracket $\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ on $\mathfrak{g}$ — by

$$\delta(t) \ := \ \frac{\Delta\big(t'\big) - \Delta^{\text{op}}\big(t'\big)}{\hbar} \ (\text{mod } \hbar) \qquad \forall\, t \in U(\mathfrak{g}) \,, \quad \begin{array}{c} t' \in U_\hbar(\mathfrak{g}) \ : \\ t' \ (\text{mod } \hbar) \, = \, t \end{array}$$

N.B.: this Poisson structure on $(G,\mathfrak{g})$ is the "semiclassical limit" or the "specialisation" of the given quantisation of the geometric object $(G,\mathfrak{g})$

[4] — The Quantum Duality Principle (=QDP) for quantum groups

Rmk: the result is due to Drinfeld — I am accountable only for proofs and terminology; other similar, loosely related claims also appeared in literature, here and there.

Theorem: ***(QDP for quantum groups — cf. [Dr], [Ga1])***

There exists an explicit **equivalence** $\big(\textbf{QUEA}\big) \underset{(\ )^\vee}{\overset{(\ )'}{\rightleftarrows}} \big(\textbf{QFSHA}\big)$ such that

$$U_\hbar(\mathfrak{g}) \mapsto U_\hbar(\mathfrak{g})' \quad \text{which is a QFSHA for the dual Poisson group } G^*$$

and

$$F_\hbar[[G]] \mapsto F_\hbar[[G]]^\vee \quad \text{which is a QUEA for the dual Lie bialgebra } \mathfrak{g}^*$$

with the functors $(\ )'$ and $(\ )^\vee$ being quasi-inverse to each other.

IDEA: at the classical level, we have two *antiequivalences*, namely

$$\text{Hopf duality} \quad U(\mathfrak{g}) \leftrightsquigarrow F[[G]] \quad \& \quad \text{Poisson duality} \quad (G,\mathfrak{g}) \leftrightsquigarrow \big(G^*,\mathfrak{g}^*\big)$$

At the quantum level, the QDP "blends together" these (classical) antiequivalences.

— REMARKS —

(1) apart from the previous **IDEA**, there is no such thing as a "semiclassical counterpart of the QDP" — in this respect, indeed,

the QDP is a "purely quantum" phenomenon.

(2) there exist several, widespread variations & consequences/applications of the QDP, e.g. to "polynomial" (rather than "formal") quantum groups (à la Jimbo & Lusztig, say), to Poisson homogeneous spaces, etc. — see [Ga2], [CiG1], [CiG2], [CFG], [EK].

(3) the QDP "behaves well" with respect to linear (Hopf) duality, in that

$$U_\hbar(\mathfrak{g}) \leftrightsquigarrow F_\hbar[[G]] \quad \Longrightarrow \quad U_\hbar(\mathfrak{g})' \leftrightsquigarrow F_\hbar[[G]]^\vee$$

that is

$$\begin{matrix} U_\hbar(\mathfrak{g}) \cong F_\hbar[[G]]^\star \\ \& \\ U_\hbar(\mathfrak{g})^* \cong F_\hbar[[G]] \end{matrix} \quad \Longrightarrow \quad \begin{matrix} U_\hbar(\mathfrak{g})' \cong \big(F_\hbar[[G]]^\vee\big)^* \\ \& \\ \big(U_\hbar(\mathfrak{g})'\big)^\star \cong F_\hbar[[G]]^\vee \end{matrix}$$

1 — **Classical setup:** Γ a Lie groupoid, $\mathcal{G} = Lie(\Gamma)$ a Lie algebroid $\Longrightarrow$

$\xrightarrow{\text{algebraically}}$ (finite projective) ***Lie-Rinehart algebra*** $\mathcal{L}$ over a *commutative* $\Bbbk$–algebra A

so $\exists \begin{cases} A \times \mathcal{L} \longrightarrow \mathcal{L} \quad \text{module structure,} \quad [\ ,\ ] : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L} \quad \Bbbk\text{–linear Lie bracket} \\ \omega : \mathcal{L} \longrightarrow Der_{\Bbbk}(A) \quad \text{"anchor" map,} \qquad \& \quad \text{compatibility axioms} \end{cases}$

then

$\mathcal{L} \xrightarrow[\text{description}]{\text{algebraic}} \begin{cases} V^{\ell}(\mathcal{L}) & \textit{left universal enveloping algebroid} \ \text{— a left bialgebroid} \\ J^{r}(\mathcal{L}) & \textit{right jet space/algebroid} \ \text{— a right bialgebroid} \end{cases}$

FACTS: *(a)* everything extends by left/right symmetry: for the same $\mathcal{L}$, there exists also $V^{r}(\mathcal{L})$, resp. $J^{\ell}(\mathcal{L})$, which is a *right* bialgebroid, resp. a *left* bialgebroid.

(b) linear duality connects $V^{\ell}(\mathcal{L}) \leftrightsquigarrow J^{r}(\mathcal{L})$ and $V^{r}(\mathcal{L}) \leftrightsquigarrow J^{\ell}(\mathcal{L})$.

Poisson structures: $\exists$ notion of "Lie-Rinehart **bi**algebra" given by either

$\mathcal{L}$ is *Lie-Rinehart algebra*		$\mathcal{L}$ is Lie-Rinehart algebra
& $\exists\ \delta : A \longrightarrow \mathcal{L}$ & $\delta : \mathcal{L} \longrightarrow \mathcal{L} \wedge \mathcal{L}$	$\xLeftrightarrow{\text{equivalent to}}$	& $\mathcal{L}^*$ is Lie-Rinehart algebra
& compatibility axioms		& compatibility axioms

$\rightarrowtail\!\!\rightsquigarrow$ ***Poisson duality:*** $\mathcal{L}$ is Lie-Rinehart **bi**algebra $\Longleftrightarrow$ $\mathcal{L}^*$ is Lie-Rinehart **bi**algebra

2 — **Quantum setup:** to $\mathcal{L}$ as above, we want to associate ***quantum groupoids***...

$V_\hbar^{\ell/r}(\mathcal{L})$ — Ping Xu introduced in [Xu] the first type of "quantum groupoid" over $\mathcal{L}$, namely ***Left Quantum Universal Enveloping AlgebroiD (=LQUEAD)***, as

$V_\hbar^\ell(\mathcal{L}) :=$ a left bialgebroid over $A_\hbar$ such that

$A_\hbar \cong A[[\hbar]]$ as topological $\Bbbk[[\hbar]]$–module, $\quad A_\hbar \big/ \hbar A_\hbar \cong A$ as $\Bbbk$–algebra

$V_\hbar^\ell(\mathcal{L}) \cong V^\ell(\mathcal{L})[[\hbar]]$ as topological $\Bbbk[[\hbar]]$–module , $\quad V_\hbar^\ell(\mathcal{L})\Big|_{\hbar=0} := V_\hbar^\ell(\mathcal{L}) \big/ \hbar V_\hbar^\ell(\mathcal{L}) \cong V^\ell(\mathcal{L})$ as a left bialgebroid over A

Remarks: *(a)* Xu also introduced *twist(or)s* for LQUEADs, and deformations by them.

(b) $\exists$ also the "right version" of this notion, namely a *right* bialgebroid $V_\hbar^r(\mathcal{L})$ over $A_\hbar$ such that [...] called "right QUEAD".

Notation: We denote by $\big(\textbf{LQUEAD}\big)_{A_\hbar}$ the category of all LQUEADs over $A_\hbar$, and by $\big(\textbf{LQUEAD}\big)$ the category of all the LQUEADs. Similarly, the categories of their "right" siblings are denoted $\big(\textbf{RQUEAD}\big)_{A_\hbar}$ and $\big(\textbf{RQUEAD}\big)$.

$J^{r/\ell}_\hbar(\mathcal{L})$ — Chemla & G. introduced in **[ChG]** the second type of "quantum groupoid" over $\mathcal{L}$, namely ***Right Quantum Formal Series AlgebroiD (=RQFSAD)***, as

$J^r_\hbar(\mathcal{L})$:= a right bialgebroid over $A_\hbar$ such that

$A_\hbar \cong A[[\hbar]]$ as topological $\Bbbk[[\hbar]]$–module, $\quad A_\hbar \big/ \hbar A_\hbar \cong A$ as a $\Bbbk$–algebra

$J^r_\hbar(\mathcal{L}) \cong J^r(\mathcal{L})[[\hbar]]$ as topological $\Bbbk[[\hbar]]$–module , $\quad J^r_\hbar(\mathcal{L})\Big|_{\hbar=0} := J^r_\hbar(\mathcal{L}) \Big/ \hbar J^r_\hbar(\mathcal{L}) \cong J^r(\mathcal{L})$ as a right bialgebroid over A

Remark & Notation: $\exists$ also the "left version", a *left* bialgebroid $J^\ell_\hbar(\mathcal{L})$ over $A_\hbar$ such that [...] called "left QFSAD". Their categories are $\big(\mathbf{RQFSAD}\big)_{A_\hbar}$ and $\big(\mathbf{LQFSAD}\big)$.

— Half-Hopf nature of quantum groupoids —

All "classical" bialgebroids $V^{\ell/r}(\mathcal{L})$ and $J^{r/\ell}(\mathcal{L})$ are actually *more*, as they are ***left and right Hopf*** *left/right bialgebroids* $\Longrightarrow$ this property is automatically inherited by any one of their quantisations $V^{\ell/r}_\hbar(\mathcal{L})$ and $J^{r/\ell}_\hbar(\mathcal{L})$ $\Longrightarrow$

in short, ***all quantum groupoids are left & right Hopf left/right bialgebroids***.

3 — Semiclassical structures from specialisation: *Quantum* $\Longrightarrow$ *Poisson*

FACT: *Every quantisation of* $\mathcal{L}$ *defines a Poisson (i.e., Lie-Rinehart* ***bi****algebra) structure on* $\mathcal{L}$ *itself*, namely:

(V) — given $V_\hbar^{\ell/r}(\mathcal{L})$, a Poisson cobracket $\delta : V^{\ell/r}(\mathcal{L}) \longrightarrow V^{\ell/r}(\mathcal{L}) \otimes V^{\ell/r}(\mathcal{L})$ is defined on $V^{\ell/r}(\mathcal{L})$ — hence a Lie cobracket $\delta : A \longrightarrow \mathcal{L}$ and $\delta : \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{L}$ — by the same recipes as for $U_\hbar(\mathfrak{g})$ in the case of QUEAs: it works the same because again (roughly) "$V_\hbar^{\ell/r}(\mathcal{L})$ is cocommutative modulo $\hbar$", see [Xu]

(J) — given $J_\hbar^{r/\ell}(\mathcal{L})$, a Poisson bracket $\{\ ,\ \} : J^{r/\ell}(\mathcal{L}) \otimes J^{r/\ell}(\mathcal{L}) \longrightarrow J^{r/\ell}(\mathcal{L})$ is defined on $J^{r/\ell}(\mathcal{L})$ by the same recipes as for $F_\hbar[[G]]$ in the case of QFSHAs: it works the same because again (roughly) "$J_\hbar^{r/\ell}(\mathcal{L})$ is commutative modulo $\hbar$", see **[ChG]**

N.B.: again, this "Poisson structure" on $\mathcal{L}$ is again called "semiclassical limit" or the "specialisation" of the given quantisation of it.

4 — Linear duality for quantum groupoids (cf. [ChG])

Quantum groupoids are defined over $A_\hbar$, possibly non-commutative $\Longrightarrow$
$\Rightarrow$ there exist (topological) ***left* dual** and ***right* dual**, that may be *different* $\Rightarrow$
$\Longrightarrow$ the theory of "(linear) dualisation" is richer and tougher.

Nevertheless, we do get what we expect, in the best possible formulation:

THEOREM 1: ***(duals of quantum groupoids — cf. [ChG])***

Let $\mathcal{L}$ be a Lie-Rinehart bialgebra, and $\mathcal{L}^{\mathrm{coop}}$ its *coopposite*.

(a) Both *right* & *left* duals of any LQUEAD for $\mathcal{L}$ are RQFSADs for $\mathcal{L}$ and $\mathcal{L}^{\mathrm{coop}}$, i.e.

$V_\hbar^\ell(\mathcal{L})^*$ is a RQFSAD for $\mathcal{L}$ & $V_\hbar^\ell(\mathcal{L})_*$ is a RQFSAD for $\mathcal{L}^{\mathrm{coop}}$

Ditto for *left* & *right* duals of any RQUEAD for $\mathcal{L}$ being LQFSADs for $\mathcal{L}$ and $\mathcal{L}^{\mathrm{coop}}$.

(b) Both *left* & *right* duals of any RQFSAD for $\mathcal{L}$ are LQUEADs for $\mathcal{L}$ and $\mathcal{L}^{\mathrm{coop}}$, i.e.

${}_\star J_\hbar^r(\mathcal{L})$ is a LQUEAD for $\mathcal{L}$ & ${}^\star J_\hbar^r(\mathcal{L})$ is a LQUEAD for $\mathcal{L}^{\mathrm{coop}}$

Ditto for *right* & *left* duals of any LQFSAD for $\mathcal{L}$ being RQUEADs for $\mathcal{L}$ and $\mathcal{L}^{\mathrm{coop}}$.

Moreover, the construction of duals is functorial, and composing twice fits well (!), hence in the end we get

COROLLARY 2: ***(antiequivalence of quantum groupoids — cf. [ChG])***

Taking left / right (topological) duals yields a bunch of ***antiequivalences***, quasi-inverse to each other, between categories of quantum groupoids of "type V" vs. "type J", e.g.

$$\big(\mathbf{LQUEAD}\big)_{A_\hbar} \underset{{}_\star(\ )}{\overset{(\ )^*}{\rightleftarrows}} \big(\mathbf{RQFSAD}\big)_{A_\hbar} \qquad \text{given by} \qquad \begin{array}{c} V_\hbar^\ell(\mathcal{L}) \mapsto V_\hbar^\ell(\mathcal{L})^* \\ {}_\star J_\hbar^r(\mathcal{L}) \leftarrow J_\hbar^r(\mathcal{L}) \end{array}$$

as well as all the sibling cases, involving the other categories.

Ditto for the larger categories when we drop the subscript "$A_\hbar$".

Remark: the situation here is quite similar to that for quantum groups, BUT for:

— the "left/right duplicity", both for the bialgebroids and for their duals, that implies that we have to keep track of and cope with a variety of objects,

— every single steps is technically much more demanding: no new ideas are needed, but to make them work is way more cumbersome.

5 — Quantum Duality Principle (=QDP) for quantum groupoids (cf. [ChG])

GOAL: find explicit **equivalences** — quasi-inverse to each other — of type

$$\big(\mathbf{LQUEAD}\big)_{A_\hbar} \underset{(\ )^\vee}{\overset{(\ )'}{\rightleftarrows}} \big(\mathbf{LQFSAD}\big)_{A_\hbar} \qquad \& \qquad \big(\mathbf{RQUEAD}\big)_{A_\hbar} \underset{(\ )^\vee}{\overset{(\ )'}{\rightleftarrows}} \big(\mathbf{RQFSAD}\big)_{A_\hbar}$$

that extend the QDP for quantum groups, in particular mapping any quantization (of either type) of $\mathcal{L}$ to a quantization (of the other type) of its dual $\mathcal{L}^*$

THEOREM 3: ***(QDP for quantum groupoids — cf. [ChG])***

There exist explicit **equivalences**, quasi-inverse to each other,

$$\big(\mathbf{LQUEAD}\big)_{A_\hbar} \underset{(\ )^\vee}{\overset{(\ )'}{\rightleftarrows}} \big(\mathbf{LQFSAD}\big)_{A_\hbar} \qquad \& \qquad \big(\mathbf{RQUEAD}\big)_{A_\hbar} \underset{(\ )^\vee}{\overset{(\ )'}{\rightleftarrows}} \big(\mathbf{RQFSAD}\big)_{A_\hbar}$$

s. t. $\quad V_\hbar^{\ell/r}(\mathcal{L})'$ is a (L/R)QFSAD for the dual Lie-Rinehart bialgebra $\mathcal{L}^*$

and

$\quad J_\hbar^{\ell/r}(\mathcal{L})^\vee$ is a (L/R)QUEAD for the dual Lie-Rinehart bialgebra $\mathcal{L}^*$

Ditto for the larger categories when we drop the subscript "$A_\hbar$".

— IDEA of the PROOF —

$\boxed{J_\hbar^{\ell/r}(\mathcal{L}) \mapsto J_\hbar^{\ell/r}(\mathcal{L})^\vee}$ — **EASY!...** In the quantum group setup, the recipe defining $F_\hbar[[G]]^\vee$ requires *multiplication* and *counit map*: both are available for quantum groupoids too, hence — up to technicalities — **the old strategy applies again**.

$\boxed{V_\hbar^{\ell/r}(\mathcal{L}) \mapsto V_\hbar^{\ell/r}(\mathcal{L})'}$ — **HARD** In the quantum group setup, the recipe defining $U_\hbar(\mathfrak{g})'$ requires *comultiplication* and *unit map* $\Longrightarrow$ the latter provides a key ingredient, namely a (direct) complement to $Ker(\epsilon)$ in $U_\hbar(\mathfrak{g})$. BUT for any $V_\hbar^{\ell/r}(\mathcal{L})$ there exists no (direct) complement to the $\left(A_\hbar \otimes A_\hbar^{\mathrm{op}}\right)$–submodule $Ker(\epsilon)$ in $V_\hbar^{\ell/r}(\mathcal{L})$...

$\Longrightarrow$ ...**we need another approach!**

IDEA: Inspired by $U_\hbar(\mathfrak{g})' = \left(F_\hbar[[G]]^\vee\right)^*$ in the quantum group case,

we **define** $V_\hbar^{\ell/r}(\mathcal{L})'$ *as "the dual" to* $J_\hbar^{\ell/r}(\mathcal{L})^\vee$ *, with* $J_\hbar^{\ell/r}(\mathcal{L}) :=$ *"dual" of* $V_\hbar^{\ell/r}(\mathcal{L})$

...YET there are *two* duals, hence we have two "candidates" for the role of $V_\hbar^{\ell/r}(\mathcal{L})'$

$\Longrightarrow$ some extra work proves that the *two "candidates" do coincide*, thus giving ONE single $V_\hbar^{\ell/r}(\mathcal{L})'$ — the rest is just skillful handicraft, stressing yet workable.

— ⋆ Final EXTRAs ⋆ —

(a) The QDP for quantum groupoids do "behave well" with respect to linear duality — yet, in the paper we did not fill in details...

(b) In the paper we also provide a concrete **example**. Namely, we consider

— $\mathfrak{g} := \Bbbk.e_1 \oplus \Bbbk.e_2$ with $\big[e_1\,,e_2\big] = e_1$ (2–dimensional, non-Abelian Lie $\Bbbk$–algebra),

— $\mathcal{L} := Der\big(S(\mathfrak{g})\big)$ as a Lie-Rinehart algebra over $A := S(\mathfrak{g})$,

— $A_\hbar := A[[\hbar]] = \big(S(\mathfrak{g})\big)[[\hbar]]$ with deformed product s.t. $e_1 \star e_2 - e_2 \star e_1 = \hbar\, e_1$,

— $V^\ell(\mathcal{L})[[\hbar]] := V^\ell\Big(Der\big(S(\mathfrak{g})\big)\Big)[[\hbar]] =$ the $\hbar$–adic completion of $V^\ell(\mathcal{L})$,

— $\mathcal{F} \in V^\ell(\mathcal{L})[[\hbar]] \,\widehat{\otimes}\, V^\ell(\mathcal{L})[[\hbar]]$ a suitable, explicit twist(or) for $V^\ell(\mathcal{L})[[\hbar]]$,

— $V^\ell_\hbar(\mathcal{L})[[\hbar]] := V^\ell(\mathcal{L})[[\hbar]]$ endowed with the left $A_\hbar$–bialgebroid structure obtained by the standard one (induced from $V^\ell(\mathcal{L})$) via *deformation by the twist(or)* $\mathcal{F}$.

For this specific example of $V^\ell_\hbar(\mathcal{L})$ — a LQUEAD which is simple enough, yet definitely non-trivial — we compute both duals $V^\ell_\hbar(\mathcal{L})_*$ and $V^\ell_\hbar(\mathcal{L})^*$, as well as $V^\ell_\hbar(\mathcal{L})'$.

— REFERENCES —

[CFG] N. Ciccoli, R. Fioresi, F. Gavarini, *Quantization of projective homogeneous spaces and duality principle*, J. Noncommut. Geom. **2** (2008), no. 4, 449–496.

[ChG] S. Chemla, F. Gavarini, *Duality functors for quantum groupoids*, J. Noncommut. Geometry **9** (2015), no. 2, 287–358.

[CiG1] N. Ciccoli, F. Gavarini, *A quantum duality principle for coisotropic subgroups and Poisson quotients*, Advances in Math. **199** (2006), no. 1, 104–135.

[CiG2] N. Ciccoli, F. Gavarini, *A global quantum duality principle for subgroups and homogeneous spaces*, Documenta Math. **19** (2014), 333–380

[Dr] V. G. Drinfeld, *Quantum groups*, Proc. ICM, Berkeley 1986 (1987), 798–820.

[EK] P. Etingof, D. Kazhdan, *Quantization of Lie bialgebras, I*, Selecta Math. (N.S.) **2** (1996), no. 1, 1–41.

[Ga1] F. Gavarini, *The quantum duality principle*, Ann. Inst. Fourier **52** (2002), no. 3, 809–834.

[Ga2] F. Gavarini, *The global quantum duality principle*, J. Reine Angew. Math. **612** (2007), 17–33.

[Xu] Ping Xu, *Quantum groupoids*, Comm. Math. Phys. **216** (2001), 539–581.